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An approach to the solution of multidimensional problems of the theory of light scattering in turbid media

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Abstract. **A new approach to developing simple analytical models in multidimensional theory of light scattering in turbid media is proposed. The approach generalises the one-dimensional Kubelka–Munk model to the cases of two and three spatial dimensions, allowing one to solve some problems explicitly with the precision that is sufficient for practical applications.**

Keywords: *light scattering, turbid media, Kubelka–Munk model.*

The widespread application of advanced laser location systems and methods of optical (laser) diagnostics to biology and medicine [1–3] revived the interest in the theoretical description of light propagation in turbid media and, in particular, the so-called transport theory (TT) [4]. Simple analytical models allowing straightforward analysis of the influence of the parameters of the medium on the behaviour of the scattered radiation field are of special interest, for example, to medical diagnostics. A special feature of such models is that they should describe the propagation of light under the conditions of strong multiple scattering. They should also yield the solution for the light flux escaping the medium through the illuminated surface, since most diagnostic techniques detect reflected (backscattered) light.

However, despite the long-term development of optical methods for light-scattering media, there are virtually no models that allow analytical solution of actual (especially multidimensional) problems. The Monte-Carlo method of statistical modelling lacks clarity and consumes a lot of computer time. The diffusion approximation, most popular in TT, lacks the required precision under the specified conditions (near the surface) and even violates proper boundary conditions [4, 5].

In this connection, many specialists devote their efforts to the improvement of existing models and analysis of their regions of applicability [5–7]. For example, interesting approaches to the improvement of small-angle approximation [7] with the help of approximating functions have been found. Recently, we have shown [8] that one-dimensional TT problems can be solved with a good precision using the well-known two-flux Kubelka–Munk (KM) models provided that the original equations are corrected at the stage of their phenomenological formulation. This approach, involving also the general problem of the ill-posed formulation of transport theory equations, can be generalised to the case of two and more spatial dimensions.

The classical TT describes the radiation inside a medium by an angle-dependent intensity whose ray derivative is defined using a priori arguments. Even in the case of two spatial dimensions, the resulting system of equations for the intensity distribution is infinitely incomplete, and as yet its explicit solution is unknown. At the same time, most practical problems require the knowledge of the entire spatial distribution of the intensity field over the bulk and surface of the medium, rather than the knowledge of the variation in the intensity along some direction x. Thus, in the case of a two-dimensional semiinfinite medium with $x > 0$ (Fig. 1), we would be interested in a three-dimensional function $R(x, y, \theta)$, which varies in the medium and characterises the intensity distribution of the radiation inside the medium.

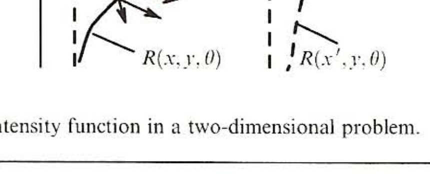

Figure 1. Intensity function in a two-dimensional problem.

When the form of this function cannot be immediately determined, the general approach of the TT is to attempt to determine the derivative of this function using 'low-level' arguments about the physics of the process. After that, the function itself can be found with the help of purely mathematical operations. Then, to find the distribution of $R(x, y, \theta)$, over the x-axis, one should specify the partial derivative $\partial R(x, y, \theta)/\partial x$ and find the required function using a trivial integration. The question is how to specify the derivative.

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The general approach of the KM method is to split the radiation field inside the medium into two counterpropagating fluxes [4]. However, according to Ref. [8], the derivative of the intensity with respect to a coordinate can be properly defined only by calculating explicitly the increment of the function over an element of length Δx and varying Δx in accordance with the conditions of the problem. In our opinion, it is reasonable to use a similar method of counterpropagating functions in the case of two or more dimensions as well (Fig. 2). Since the main difficulty arises from specifying the derivative rather than solving the equations, we will consider only one of the fluxes, R_+ for example, and ignore R_- for the moment to illustrate the proposed approach. According to the definition of a partial derivative,

$$\frac{\partial R_+(x,y',\theta)}{\partial x}=\lim_{\Delta x\to 0}\frac{R_+(x+\Delta x,y',\theta)-R_+(x,y',\theta)}{\Delta x}. \tag{1}$$

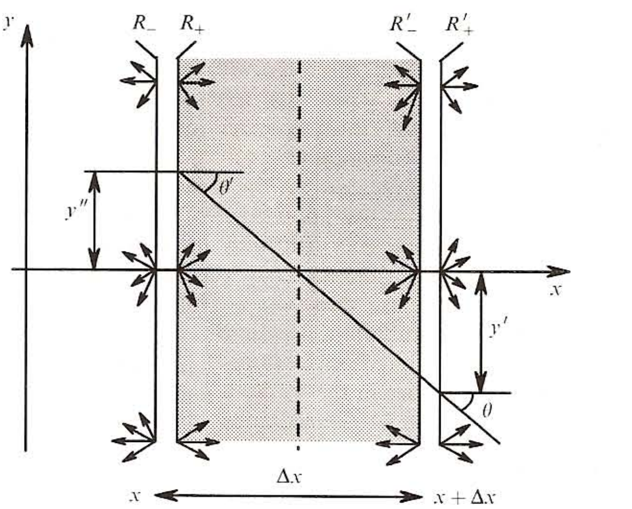

Figure 2. Method of counterpropagating functions in multidimensional problems.

Following the method for defining the derivative used in Ref. [8] and taking into account that the absorption and scattering events are separated in space (Fig. 2), we write

$$R_+(x+\Delta x,y',\theta)=\int_{-\pi/2}^{\pi/2}R_+(x,y'',\theta')\mathrm{e}^{-k\Delta x/\cos\theta'}\rho(\theta,\theta')\mathrm{d}\theta', \tag{2}$$

where k is the linear absorption coefficient of the medium; ρ is the phase (angular) function of scattering; and variable y'' can be represented as

$$y''=y'+\Delta x\tan\theta'. \tag{3}$$

Since the physics of the process is invariant along the coordinate x, the function $R_+(x,y,\theta)$ appearing in Eqn (1) obviously satisfies the following integral equality:

$$R_+(x,y',\theta)=\int_{-\pi/2}^{\pi/2}R_+(x,y',\theta')\rho(\theta,\theta')\mathrm{d}\theta'. \tag{4}$$

Then, inserting expression (3) into Eqn (2) and further to Eqn (1) and inserting (4) to Eqn (1), we obtain an indeterminacy of the 0/0 type at $\Delta x\to 0$. Removing the indeterminacy with the help of the L'Hospital rule and using the fact that the limit of a function equals the function of a limit [9], we obtain for $R_+(x,y',\theta)$ the required first-order partial differential equation:

$$\frac{\partial R_+(x,y',\theta)}{\partial x}=\int_{-\pi/2}^{\pi/2}\left[\frac{\partial R_+(x,y',\theta')}{\partial y}\tan\theta'\right.$$
$$\left.-\frac{k}{\cos\theta'}R_+(x,y',\theta')\right]\rho(\theta,\theta')\mathrm{d}\theta'. \tag{5}$$

If we take flux $R_-(x,y,\theta)$, in consideration, an integral of $R_-(x,y,\theta)$ appears in the right-hand side of Eqn (5), similarly to the KM method. In addition, the general equation (5) should be complemented by a coupled integro-differential equation for the flux $R_-(x,y,\theta)$. In other words, in the general case of a multidimensional problem, one should solve a system of two integro-differential equations containing first partial derivatives.

In our opinion, the approach based on equation (5) is remarkable in two ways. First, the radiation field $R(x,y,\theta)$ in the two-dimensional space is described by a function that does not allow separation of variables. Note that today the general equations of mathematical physics are solved with the help of the Fourier method (separation of variables). Our approach makes it possible to describe radiation fields by functions whose variables do not separate. Second, the use of equation (5) implies the appearance of an explicit multidimensional function of boundary conditions at $x=0$, function $R_0(y,\theta)$.

Concerning the methods for solving multidimensional problems of TT that are based on a system of linear equations for discrete one-dimensional directions (e.g., the method of discrete ordinates), note that they involve boundary conditions in the form of discrete numbers for each direction, which alters the dimensionality of the problem. In addition, these discrete numbers are sometimes unknown when the problem is stated. In our case, the boundary conditions are formulated most naturally in the form of a distributed function.

The validity of the proposed approach can be demonstrated, for example, in the simplest case of a homogeneous nonscattering medium. The absence of scattering will result in the disappearance of the integral from the original equation (2). Equation (5) then reads

$$\frac{\partial R_+(x,y',\theta)}{\partial x}=\frac{\partial R_+(x,y',\theta)}{\partial y}\tan\theta-\frac{k}{\cos\theta}R_+(x,y',\theta). \tag{6}$$

which is a quasi-linear equation of the type [9,10]:

$$a\frac{\partial F}{\partial x}+b\frac{\partial F}{\partial y}=cF. \tag{7}$$

The Monge cone for this equation degenerates to a Monge axis, so that each of its points corresponds to a single characteristic direction [9]. Therefore, the solution of equation (6) should not pose any serious difficulties (equations of type (7) have tabulated solutions [10]). In the special case of a $x>0$ semispace that is uniformly illuminated by flux $R_0(x=0,y,\theta)=R_0(\theta)$, the solution is independent of the coordinate y and is given by

$$R(x,\theta)=R_0(\theta)\mathrm{e}^{-kx/\cos\theta}. \tag{8}$$

One can easily see that in the case of $k = 0$, the original flux will propagate unchanged in the medium, as indeed it should. In the case of a one-dimensional problem, we have $\theta = 0$, and formula (8) reduces to the trivial Bouguer law, also in agreement with the logic of the problem.

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